

CHARACTERIZATION OF DISTRIBUTIVITY IN A SOLID

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ABSTRACT. We give a characterization of the validity of the distributive law in a solid. There exists equivalence between the characterization and the modified axiom of distributivity valid in a solid.

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1. INTRODUCTION

Solids arise as extensions of fields [2], typically non-archimedean fields or the nonstandard reals [4][5], in the form of cosets with respect to convex subgroups. Such convex subgroups may be seen as orders of magnitude and are called *magnitudes*. In solids the laws of addition and multiplication are more those of completely regular semigroups [3][6] than of proper groups. Also the distributive law is not valid in general, but there exists an adapted form of distributivity, introducing a correction term in the form of a magnitude. In this paper we characterize the validity of the ordinary distributive law in a solid (Theorem 4.2). Let x, y, z be arbitrary elements in a solid. The conditions of the characterization given in this paper roughly state that in order for distributivity to fail the factor x should be more imprecise than the terms y and z , and these terms should be almost opposite. Special cases where distributivity does hold include magnitudes, elements of the same sign, and precise elements (elements with minimal magnitude).

The equality expressed by the adapted distributivity axiom of [2] (Axiom 22 of the Appendix) and the characterization of distributivity by Theorem 4.2 are shown to be equivalent.

The results require a thorough investigation of the properties of magnitudes and precise elements. This is done in Section 2 and 3. In Subsection 4.1 we state necessary and sufficient conditions for distributivity to hold. In Subsection 4.2 we prove that the equivalency of these conditions to proper distributivity is equivalent to the distributivity law with correction term, as given by Axiom 22.

For notation and terminology we refer to [2]. For the sake of reference a complete list of axioms is given in the Appendix. We recall that, given an element x of a solid, its individualized neutral element e (see Axiom 3) as such is unique, and has the functional notation $e = e(x)$. In the same way the symmetric element s of Axiom 3 is denoted by $s(x) \equiv -x$, the individualized unity u of Axiom 8 is denoted by $u(x)$ and the multiplicative inverse d of Axiom 9 is denoted by $d(x) \equiv x^{-1} \equiv /x$.

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2. ALGEBRAIC PROPERTIES OF MAGNITUDES

In this section we study algebraic properties of magnitudes in an assembly. Assemblies were introduced in [1]. The results in this section will be used to prove the characterization of distributivity in a solid.

2.1. Neutral and symmetric elements. We verify some elementary properties of magnitudes and symmetrical elements. Part of it are generalizations of the usual properties of neutral and symmetric elements and others deal with their functional representation.

We start by recalling some results on magnitudes of [1].

Theorem 2.1 ([1, Thm 4.11]). (Cancellation law) *Let A be an assembly and let $x, y, z \in A$. Then $x + y = x + z$ if and only if $e(x) + y = e(x) + z$.*

Proposition 2.2 ([1, Thm 4.12]). *Let A be an assembly. Then for all $x, y \in A$*

- (1) (Idempotency for addition) $e(x) + e(x) = e(x)$.
- (2) (Linearity of e) $e(x + y) = e(x) + e(y)$.
- (3) (Absorption) $e(x) + e(y) = e(x)$ or $e(x) + e(y) = e(y)$.
- (4) (Idempotency for composition) Let A be an assembly and let $x \in A$. Then $e(e(x)) = e(x)$.

As a consequence a magnitude can only be the magnitude of itself.

Theorem 2.3. (Representation) *Let A be an assembly and let $x, y \in A$. If $x = e(y)$ then $x = e(x)$.*

Proof. Suppose $y \in A$ is such that $x = e(y)$. Then $e(x) = e(e(y)) = e(y) = x$ by Proposition 2.2.4. \square

As regards to the symmetric function, it is easy to see that it is injective. We verify now that it is linear with respect to addition and has the symmetry property, meaning that the inverse of the inverse of a given element is the element itself. We also show that the composition of the inverse function with the neutral function is equal to the neutral function.

Proposition 2.4. *Let A be an assembly and let $x \in A$. Then*

- (1) $-(-x) = x$.
- (2) $-(x + y) = -x - y$.
- (3) $e(-x) = -e(x) = e(x)$.

Proof. 1. By Axiom 4 one has $e(-(-x)) = e(-x) = e(x) = -x + x$. Hence $-(-x) = -(-x) + e(-(-x)) = -(-x) - x + x = e(-x) + x = e(x) + x = x$.

2. By Proposition 2.2.2 and Axiom 4 one has $-(x + y) + x + y = e(x + y) = e(x) + e(y) = -x + x - y + y = -x - y + x + y$. Then by the cancellation law $-(x + y) + e(x + y) = -x - y + e(x + y)$. Again using the linearity of e and Axiom 4 one obtains

$$\begin{aligned} -(x + y) + e(-(x + y)) &= -x - y + e(-x) + e(-y) \\ &= -x + e(-x) - y + e(-y). \end{aligned}$$

Hence $-(x + y) = -x - y$.

3. By Axiom 4 one only has to show that $-e(x) = e(x)$. Using Proposition 2.2.1 and Proposition 2.2.2 one derives $e(x) = -x + x = -x - (-x) = -(x - x) = -e(x)$. \square

2.2. Order and the magnitudes.

Definition 2.5. We say that an assembly A is an ordered assembly if it satisfies the axioms of order 11-18.

As far as magnitudes are concerned, the order relation can be defined in terms of addition and corresponds to the natural order relation in semigroups [3][6].

Proposition 2.6. Let A be an ordered assembly. Let $x, y \in A$. Then $e(x) + e(y) = e(x)$ if and only if $e(y) \leq e(x)$.

Proof. By Axiom 16 we only need to prove the necessary part. Suppose that $e(y) \leq e(x)$. Now $e(x) + e(y) = e(x)$ or $e(x) + e(y) = e(y)$ by Proposition 2.2.2. If $e(x) + e(y) = e(x)$, there is nothing to prove. If $e(x) + e(y) = e(y)$, by Axiom 16, $e(x) \leq e(y)$. But then, by antisymmetry, $e(x) = e(y)$. Hence $e(x) + e(y) = e(x) + e(x) = e(x)$. \square

We say that x is *positive* if $e(x) \leq x$ and *negative* if $x < e(x)$. With these notions it is possible to define an absolute value.

Definition 2.7. Let A be an ordered assembly and let $x \in A$. The absolute value of x is defined as

$$|x| \equiv \begin{cases} x, & \text{if } e(x) \leq x \\ -x, & \text{if } x < e(x). \end{cases}$$

It follows from compatibility with addition that $e(x) \leq x$ if and only if $-x \leq e(x)$. We show that the sum of two positive elements is also positive and that every element which is larger than or equal to a positive element is also positive. Then we prove that Proposition 2.6 may be extended to any positive element. We finish with some strict inequalities.

Proposition 2.8. Let A be an ordered assembly and let $x, y \in A$.

- (1) If x and y are both positive then $x + y$ is also positive.
- (2) If $e(y) \leq y \leq x$ then $e(x) \leq x$.

Proof. 1. If $e(x) \leq x$ and $e(y) \leq y$, then $e(x + y) = e(x) + e(y) \leq x + y$.

2. Assume that $e(y) \leq y \leq x$. If $e(y) \leq e(x)$ then $e(x) + e(y) = e(x)$ by Proposition 2.6. By compatibility with addition $e(x) = e(x) + e(y) \leq e(x) + x = x$. If $e(x) \leq e(y)$, by transitivity $e(x) \leq x$. \square

The next theorem states that a positive number leaves a magnitude invariant if and only if it is smaller than this magnitude.

Theorem 2.9. Let A be an ordered assembly and let $x, y \in A$. If y is positive then $y \leq e(x)$ if and only if $e(x) + y = e(x)$.

Proof. By Axiom 16 we only need to prove the sufficiency. Assume that y is positive and that $y \leq e(x)$. Then by transitivity $e(y) \leq e(x)$ and $e(x) + e(y) = e(x)$ by Proposition 2.6. By compatibility with addition $e(x) = e(x) + e(y) \leq e(x) + y \leq e(x) + e(x) = e(x)$. Hence $e(x) = e(x) + y$. \square

Proposition 2.10. Let A be an ordered assembly and let $x, y, z \in A$.

- (1) If $x < e(x)$ and $e(y) < y$, then $x < y$.
- (2) If $x < e(x)$, then $x < e(y)$.

(3) If $x < y + e(z)$ and $e(z) < e(x)$, then $x < y$.

Proof. 1. Assume $x < e(x)$ and $e(y) < y$. If $e(x) \leq e(y)$ the result follows by transitivity. If $e(y) < e(x)$ then $e(x) = e(y) + e(x)$ by Proposition 2.6. Suppose, towards a contradiction, that $y \leq x$. Then $e(x) = e(y) + e(x) \leq y + e(x) \leq x + e(x) = x$, a contradiction.

2. Assume $x < e(x)$. We suppose towards a contradiction $e(y) \leq x$. Then $e(y) < e(x)$ and $e(x) = e(y) + e(x)$ by Proposition 2.6. Hence $e(x) = e(y) + e(x) \leq x + e(x) = x$, a contradiction.

3. Assume $x < y + e(z)$ and $e(z) < e(x)$. By Proposition 2.6, one has $e(z) + e(x) = e(x)$. Suppose towards a contradiction that $y \leq x$. Then $y + e(z) \leq x + e(z) = x$, a contradiction. \square

2.3. Magnitudes and the product. We denote by S^* the set of all elements of S which are not magnitudes, i.e. $S^* = \{x \in S \mid x \neq e(x)\}$.

The following lemma shows that unity elements of zeroless elements are zeroless, implying that magnitudes and unities in a solid are distinct.

Lemma 2.11. *Let S be a solid and let $x, y \in S^*$*

- (1) $u(x) \neq e(u(x))$.
- (2) $u(x) \neq e(y)$.

Proof. 1. Suppose that $u(x) = e(u(x))$. Then $x = xu(x) = xe(u(x))$. By Axiom 19 there is z such that $x = e(z)$ and by Theorem 2.3, $x = e(x)$. Hence, if $x \neq e(x)$, then $u(x) \neq e(u(x))$.

2. Suppose towards a contradiction that $u(x) = e(y)$ for some $y \in A$. Then $u(x) = e(u(x))$ by Theorem 2.3, in contradiction with Part 1. \square

As a consequence of the previous lemma multiplicative inverses of zeroless elements are zeroless. Indeed, if $x^{-1} \neq e(x^{-1})$, one must have $u(x) = xx^{-1} = e(x^{-1})$. Then $u(x)$ would be a magnitude by Axiom 19, a contradiction.

Let S be a solid and let $x, y \in S^*$. Concerning the magnitude of the product, one should expect that

$$\begin{aligned} (1) \quad e(xy) &= e((x + e(x))(y + e(y))) \\ &= e(xy + e(x)y + e(y)x + e(x)e(y)) \\ &= e(x)y + e(y)x + e(x)e(y). \end{aligned}$$

Axiom 20 states that we may neglect the last term. In the next subsection, using the order relation, we show that the term $e(x)e(y)$ is less indeed than both $e(x)y$ and $e(y)x$.

Here we present some usefull properties of multiplication by magnitudes obtained by mere algebraic methods.

If one of the factors of the product is a magnitude, we have the following simplification.

Proposition 2.12. *Let S be a solid and let $x, y \in S$. If $x = e(x)$ then $e(xy) = e(x)y$.*

Proof. By Axiom 19, $e(x)y$ is a magnitude. Then $e(xy) = e(e(x)y) = e(x)y$. \square

Whenever one multiplies a given element by a magnitude, the sign of that element can be neglected.

Proposition 2.13. *Let S be a solid and let $x, y \in S$. Then $e(y)(-x) = e(y)x$.*

Proof. By Axiom 23 and Proposition 2.4.3 one has $e(y)(-x) = -(e(y)x) = (-e(y))x = e(y)x$. \square

In integral domains the product of two non-zero elements is always non-zero. In solids the product of two zeroless elements is always zeroless.

Theorem 2.14. *Let S be a solid and let $x, y \in S$. Then $xy = e(xy)$ if and only if $x = e(x)$ or $y = e(y)$.*

Proof. Suppose firstly that $x = e(x)$ or $y = e(y)$. If $x = e(x)$, then $xy = e(x)y$ and by Axiom 19 there is z such that $xy = e(z)$. By Theorem 2.3 we conclude that $xy = e(xy)$. If $y = e(y)$ the proof is analogous.

Secondly, suppose towards a contradiction that $xy = e(xy)$, while $x \neq e(x)$ and $y \neq e(y)$. Without loss of generality we may assume that $u(x)u(y) = u(x)$. Then

$$u(x) = u(x)u(y) = xx^{-1}yy^{-1} = e(xy)x^{-1}y^{-1}.$$

By Axiom 19 there is z such that $e(z) = e(xy)x^{-1}y^{-1}$, in contradiccion with Lemma 2.11. Hence if $xy = e(xy)$, then $x = e(x)$ or $y = e(y)$. \square

As a corollary we obtain that if an element is not a magnitude, its square (element multiplied by itself) is also not a magnitude.

Corollary 2.15. *Let S be a solid and let $x \in S^*$, then $x^2 \neq e(x^2)$.*

In rings one has that $1 \cdot 0 = 0$. Next theorem generalizes this to solids. A further generalization will be given in Theorem 4.15 below. As a consequence we obtain an expression for the magnitude of the inverse.

Theorem 2.16. *Let S be a solid. Suppose $x \in S^*$. Then*

- (1) $u(x)e(x) = xe(u(x)) = e(x)$.
- (2) $e(x^{-1}) = e(x)x^{-1}x^{-1}$.

Proof. 1. Suppose that $x \neq e(x)$. Then $e(x) = e(xu(x)) = e(x)u(x) + xe(u(x))$ by Axiom 8 and Axiom 20. Hence applying Axiom 21

$$e(x) = e(x)u(x) + xe(x)x^{-1} = e(x)u(x) + e(x)u(x) = e(x)u(x).$$

This implies that

$$xe(u(x)) = xe(x)x^{-1} = e(x)u(x) = e(x).$$

2. Note that by Axiom 21

$$e(x^{-1})x = e(x^{-1})(x^{-1})^{-1} = e(u(x^{-1})) = e(u(x)) = e(x)x^{-1}.$$

Hence

$$e(x^{-1}) = e(x^{-1})u(x^{-1}) = e(x^{-1})u(x) = e(x^{-1})xx^{-1} = e(x)x^{-1}x^{-1}.$$

\square

2.4. Order and the product. We first investigate how the order behaves under multiplication by magnitudes. As a result we obtain that the product of two positive numbers is still positive, implying that squares are positive and a fortiori the unity. Inequalities are reversed when multiplying by a negative element upon the application of a correction term. The rules of the sign of the inverse are as in rings.

Next proposition states that the order relation is preserved under scaling, and Lemma 2.18 and Proposition 2.19 indicate that the neglect of the term $e(x)e(y)$ in the formula for the magnitude of the product of Axiom 20 is justified, as announced in the previous subsection.

Proposition 2.17. *Let S be a solid and let $x, y, z \in S$. If $e(y) \leq e(z)$ then $xe(y) \leq xe(z)$.*

Proof. If $e(x) = x$ the result follows by Axiom 18. If $e(x) < x$, by Axiom 17 one has $xe(y) \leq xe(z)$. If $x < e(x)$, then $e(-x) = e(x) < -x$. Hence $-xe(y) \leq -xe(z)$ by compatibility with multiplication. By Proposition 2.13 one has $xe(y) \leq xe(z)$. \square

Lemma 2.18. *Let S be a solid and let $x, y \in S$. Then $e(x)e(y) \leq xe(y)$.*

Proof. If $e(x) = x$ there is nothing to show. If $e(x) < x$, the result follows by compatibility with multiplication because $e(y) \leq e(y)$. If $x < e(x)$, then $e(x) < -x$. Then $e(x)e(y) \leq -xe(y) = xe(y)$ by Proposition 2.13 and Axiom 18. \square

Proposition 2.19. *Let S be a solid and let $x, y \in S$. Then $e(x)e(y) \leq e(xy)$.*

Proof. By Lemma 2.18 it holds that $e(x)e(y) \leq xe(y)$ and $e(x)e(y) \leq ye(x)$. Hence using Axiom 20,

$$e(x)e(y) \leq xe(y) + ye(x) = e(xy).$$

\square

We are now able to prove that products of two positive elements and squares are positive.

Theorem 2.20. *Let S be a solid and let $x, y \in S$. If x and y are both positive then $e(xy) \leq xy$.*

Proof. Suppose x and y are both positive. If $y = e(y)$ or $x = e(x)$ by Axiom 19 and Theorem 2.3 one has $e(xy) = xy$. If $e(x) < x$ and $e(y) < y$, by Axiom 17, $xe(y) \leq xy$ and $ye(x) \leq xy$. Then, using Axiom 20,

$$e(xy) = xe(y) + ye(x) \leq xy + xy.$$

By adding $-(xy)$ to both sides of the equation one obtains $-xy \leq xy$. Then $e(xy) \leq xy$. \square

Corollary 2.21. *Let S be a solid and let $x \in S^*$. Then*

- (1) $e(x^2) \leq x^2$. Moreover, equality holds if and only if $x = e(x)$.
- (2) $e(u(x)) < u(x)$.

Proof. 1. We show firstly that $e(x^2) \leq x^2$. If x is positive, the result follows from Theorem 2.20. If x is negative, then $-x$ is positive. Hence

$$e(x^2) = e((-x)^2) \leq (-x)^2 = x^2.$$

Secondly, if $x = e(x)$, Proposition 2.19 implies that $x^2 = x^2 + e(x^2) = e(x)^2 + e(x^2) = e(x^2)$. If $x \neq e(x)$, by Corollary 2.15 one has $x^2 \neq e(x^2)$. Hence equality holds if and only if $x = e(x)$.

2. By Part 1 because $u(x)u(x) = u(x)$. \square

In ordered rings inequalities are reversed upon multiplying by a negative element. This remains true for zeroless elements.

Proposition 2.22. *Let S be a solid and let $x, y \in S$. If $e(x) < x$ and $y < e(y)$ then $xy < e(xy)$.*

Proof. Suppose $e(x) < x$ and $y < e(y)$. Then $e(y) < -y$, hence $e(x)e(y) \leq x(-y)$. Then by Proposition 2.19 and compatibility with addition

$$xy = xy + e(x)e(y) \leq xy + x(-y) = xy - xy = e(xy).$$

Because $x \neq e(x)$ and $y \neq e(y)$ one has $xy \neq e(xy)$, by Theorem 2.14. Hence $xy < e(xy)$. \square

In the general case one must take into account the magnitudes of the products of the elements.

Proposition 2.23. *Let S be a solid and let $x, y, z \in S$. Suppose that $y \leq z$. If $x < e(x)$ then $xz + e(xy) \leq xy + e(xz)$.*

Proof. Suppose that $y \leq z$. If $x < e(x)$, then $e(-x) = e(x) < -x$. By compatibility with multiplication $(-x)y \leq (-x)z$. Then $-(xy) \leq -(xz)$ by Axiom 23. Applying Axiom 15 twice, we see that $xz + e(xy) \leq xy + e(xz)$. \square

Within ordered rings with unity it holds that $0 < 1$, the inverse for multiplication of a positive element is positive, and the inverse of an element larger than 1 is smaller than 1. We adapt these properties to solids.

Proposition 2.24. *Let S be a solid and let $x \in S^*$. Then $e(x) < x$ if and only if $e(x^{-1}) < x^{-1}$.*

Proof. Assume that $e(x) < x$. Because $x \neq e(x)$, by Lemma 2.11.2 one has $x^{-1} \neq e(x^{-1})$. By Theorem 2.14 one has $x^{-1}x^{-1} \neq e(x^{-1}x^{-1})$. Then by compatibility with multiplication and Corollary 2.21.1

$$xe(x^{-1}x^{-1}) \leq xx^{-1}x^{-1} = u(x)x^{-1} = x^{-1}.$$

Then by Theorem 2.16.1 and Axiom 20

$$\begin{aligned} e(x^{-1}) &= u(x^{-1})e(x^{-1}) \\ &= u(x)e(x^{-1}) \\ &= xe(x^{-1})x^{-1} \\ &= x(e(x^{-1})x^{-1} + x^{-1}e(x^{-1})) \\ &= xe(x^{-1}x^{-1}) \leq x^{-1}. \end{aligned}$$

Hence $e(x^{-1}) < x^{-1}$.

Conversely, assume that $e(x^{-1}) < x^{-1}$. By the above $e(x) = e((x^{-1})^{-1}) < (x^{-1})^{-1} = x$. \square

Corollary 2.25. *Let S be a solid and let $x \in S^*$. If $u(x) \leq x$ then $e(x) < x$.*

Proof. By Proposition 2.8.2, with $y = u(x)$, which is positive by Corollary 2.21.2. \square

Proposition 2.26. *Let S be a solid and let $x \in S^*$. Then*

- (1) *If $u(x) \leq x$, then $x^{-1} \leq u(x)$.*
- (2) *If $e(x) < x$ and $x \leq u(x)$, then $u(x) \leq x^{-1}$.*

Proof. 1. Suppose that $u(x) \leq x$. Then by Corollary 2.25 one has $e(x) < x$ and then $e(x^{-1}) < x^{-1}$ by Proposition 2.24. Hence Axiom 17 implies that $x^{-1} = x^{-1}u(x) \leq x^{-1}x = u(x)$.

2. Suppose that $e(x) < x$ and $x \leq u(x)$. Then $e(x^{-1}) < x^{-1}$ by Proposition 2.24. Again Axiom 17 implies that $u(x) = xx^{-1} \leq u(x)x^{-1} = u(x^{-1})x^{-1} = x^{-1}$. \square

Finally we show that if a positive element is larger than its inverse, its magnitude must be larger than the magnitude of the inverse.

Proposition 2.27. *Let S be a solid and let $x \in S^*$. If $e(x^{-1}) < x^{-1} \leq x$ then $e(x^{-1}) \leq e(x)$.*

Proof. Suppose $e(x^{-1}) < x^{-1} \leq x$. Then $x^{-1}x^{-1} \leq xx^{-1} = u(x)$, by Axiom 17. By Theorem 2.16 and Axiom 18 one has

$$e(x^{-1}) = x^{-1}x^{-1}e(x) \leq u(x)e(x) = e(x).$$

\square

2.5. Distributivity with magnitudes. In this subsection we prove distributivity in some special cases, without using Axiom 22. We show that the distributive property $x(y+z) = xy+xz$ holds in case y and z are both magnitudes, and in case one of those is a magnitude, less than or equal to the magnitude of the remaining element. In particular one always has $x(y+e(y)) = xy+xe(y)$.

Proposition 2.28. *Let S be a solid and let $x, y, z \in S$. Then $x(e(y)+e(z)) = xe(y)+xe(z)$.*

Proof. We may suppose without loss of generality that $e(z) \leq e(y)$. Then $e(y)+e(z) = e(y)$ by Proposition 2.6. By Proposition 2.17 one has $e(z)x \leq e(y)x$. Then Axiom 19 and Proposition 2.6 imply that $e(y)x + e(z)x = e(y)x$.

Hence

$$x(e(y)+e(z)) = xe(y) = xe(y)+xe(z).$$

\square

As a consequence of the previous proposition distributivity holds for the magnitudes.

Corollary 2.29. *Let S be a solid and let $x, y, z \in S$. Then $e(x)(e(y)+e(z)) = e(x)e(y)+e(x)e(z)$.*

Proposition 2.30. *Let S be a solid and let $x, y, z \in A$. If $e(z) \leq e(y)$, then $x(y+e(z)) = xy+xe(z)$.*

Proof. Assume that $e(z) \leq e(y)$. Then by Proposition 2.6 one has $x(y + e(z)) = x(y + e(y) + e(z)) = xy$. Now $xe(z) \leq xe(y)$ by Proposition 2.17 and $xe(y) + xe(z) = xe(y)$ by Proposition 2.6. Then Axiom 20 implies that

$$\begin{aligned} xy + xe(z) &= xy + e(xy) + xe(z) \\ &= xy + e(x)y + xe(y) + xe(z) \\ &= xy + e(x)y + xe(y) = xy + e(xy) = xy. \end{aligned}$$

Hence $x(y + e(z)) = xy + xe(z)$. \square

Corollary 2.31. *Let S be a solid and let $x, y \in S$. Then $xy = x(y + e(y)) = xy + xe(y)$.*

3. THE FIELD OF PRECISE ELEMENTS

Definition 3.1. *Let S be a solid. An element $x \in S$ is said to be precise if $e(x) = m$.*

The elements m and u given by Axioms 24 and 25 in solids are unique. We call them zero and one respectively. We prove that zero and one have the expected properties concerning the order relation and multiplication. The precise elements are closed under addition, multiplication and inversion, in fact constitute a field.

The proof of the fact that the elements m and u are unique is identical as for groups and will be omitted.

Proposition 3.2. *Let S be a solid. There is exactly one element m and exactly one element u such that $x + m = x$ and $xu = x$ for all $x \in S$.*

Definition 3.3. *We call zero the unique element m such that $x + m = x$ for all $x \in S$, and it will be denoted by 0. We call one the unique element u such that $xu = x$ for all $x \in S$, and it will be denoted by 1.*

In the following proposition we show some elementary properties of the elements 0 and 1.

Proposition 3.4. *Let S be a solid and $x, y, z \in S$. Then*

- (1) $e(0) = 0$.
- (2) $e(1) = 0$.
- (3) $1 \neq 0$.
- (4) $u(1) = 1$.
- (5) $-0 = 0$.
- (6) $1 \cdot 0 = 0$.
- (7) $0 < 1$.
- (8) $0 \leq e(x)$.
- (9) If $0 \leq x$ then $e(x) \leq x$.
- (10) If $0 \leq x$ and $0 \leq y$, then $0 \leq x + y$.
- (11) If $x \leq 0$ and $y \leq 0$, then $x + y \leq 0$.

Proof. 1. By Axiom 3 and 24 one has $0 = 0 + e(0) = e(0)$.

2. Let $x \in S$. Then $x(1 + e(1)) = x1 = x$. By Corollary 2.31

$$x(1 + e(1)) = x1 + xe(1) = x + xe(1).$$

Hence $x = x + xe(1)$. Since x is arbitrary, by Proposition 3.2 one has $xe(1) = 0$. Putting $x = 1$, one obtains $1e(1) = 0$. Since $1e(1) = e(1)$ by Axiom 25, one derives that $e(1) = 0$.

3. Because S is an solid, it has a zeroless element, say x . If $1 = 0$, then $x = x \cdot 1 = x \cdot 0 = x \cdot e(0)$, so by Axiom 19 x is a magnitude, a contradiction. Hence $1 \neq 0$.

4. By Axiom 8 and 25 one has $1 = 1 \cdot u(1) = u(1)$.

5. Using Part 1 and Proposition 2.4.3 one has $-0 = -e(0) = e(0) = 0$.

6. By Part 4, Part 2 and Theorem 2.16.1.

7. Corollary 2.21.2, Part 2 and Part 4 imply that $0 = e(1) = e(u(1)) < u(1) = 1$.

8. Directly from Axiom 24 and Axiom 16.

9. Suppose that $0 \leq x$. Then $e(x) = 0 + e(x) \leq x + e(x) = x$ by compatibility with addition.

10. Directly from Axiom 15.

11. Directly from Axiom 15. □

We prove that zero is the absorbing element for multiplication.

Proposition 3.5. *Let S be a solid. Then $x0 = 0$, for all $x \in S$.*

Proof. By Proposition 3.4.2 and Corollary 2.31 one has

$$x(1 + 0) = x1 + x0$$

for all $x \in S$. Hence $x = x + x0$ for all $x \in S$. Then $x0 = 0$ for all $x \in S$ by Proposition 3.2. □

By Proposition 3.4.1 and 3.4.2 the elements 0 and 1 are examples of precise elements. We show that precise elements are closed under elementary operations.

Proposition 3.6. *Let S be a solid and let $a, b \in S$ be precise. Then $a + b$, $-b$ and ab are precise. If $a \neq 0$ then $u(a)$ and a^{-1} are also precise.*

Proof. Since a and b are precise, one has $e(a) = e(b) = 0$. Then $e(-b) = e(b) = 0$ and $e(a + b) = e(a) + e(b) = 0$. Hence $-b$ and $a + b$ are precise. Also, by Axiom 20 and Proposition 3.5

$$e(ab) = e(a)b + ae(b) = 0.$$

Hence ab is precise.

Suppose now that $a \neq 0$. Then by Axiom 21 and Proposition 3.5

$$e(u(a)) = e(a)a^{-1} = 0a^{-1} = 0.$$

Hence $u(a)$ is precise. Finally, by Theorem 2.16.2 and Proposition 3.5

$$e(a^{-1}) = e(a)a^{-1}a^{-1} = 0a^{-1}a^{-1} = 0.$$

Hence a^{-1} is also precise. □

Proposition 3.7. *Let S be a solid, let $x, y \in S$ and let $a \in S$ be precise. Then $a(x + y) = ax + ay$.*

Proof. From Axiom 22 and Proposition 3.5 we derive

$$\begin{aligned} ax + ay &= a(x + y) + e(a)x + e(a)y \\ &= a(x + y) + 0x + 0y = \\ &= a(x + y). \end{aligned}$$

□

Theorem 3.8. *The set of precise elements P of a solid S is an ordered field.*

Proof. By Proposition 3.6, precise elements are closed under addition and multiplication. By Proposition 3.4.3 the precise element 0 is different from the precise element 1. Clearly $(P, +)$ and $(P \setminus \{0\}, \cdot)$ are ordered abelian groups. The distributive property holds by Proposition 3.7. Hence P is an ordered field. □

Corollary 3.9. *Let S be a solid and let $a \in S$ be a non-zero precise element. Then $u(a) = 1$.*

4. CHARACTERIZATION OF DISTRIBUTIVITY

In solids addition is connected with multiplication via an adapted distributivity condition. Distributivity holds up to a magnitude. Indeed, if x, y and z are elements of a solid, in general to obtain equality with $xy + xz$ one has to add to $x(y + z)$ a magnitude depending only on y, z and the magnitude of x . So we have subdistributivity with a concrete expression of the correction term.

Theorem 4.2 gives necessary and sufficient conditions for the usual distributive law to hold, i.e. under such conditions the correction term is less than the magnitudes involved. It appears that the only case where distributivity does not hold is the joint presence of two circumstances: the factor x should be more imprecise than the terms y and z , and these terms should be almost opposite. That approximations may not work out well in relation to differences of almost equal terms is well-known, and may for instance be compared with the fact that if a sequence of functions f converges, the corresponding sequence of derivatives (limits involving differences with almost equal terms of the form $f(x + h) - f(x)$) does not need to converge.

4.1. Conditions for distributivity. We start by formulating necessary and sufficient conditions for distributivity to hold. This requires a notion of relative size of magnitude which corresponds to the notion given in [5, p. 151] (see also [4, Definition 3.2.11]).

Definition 4.1. *Let S be a solid and $x \in S$. The relative uncertainty of x , noted $R(x)$ is defined as follows. If $x \neq e(x)$, then $R(x) = e(u(x))$. If $x = e(x)$, then $R(x) = M$, where M is given by Axiom 26.*

Observe that for zeroless x one has $R(x) = e(x)x^{-1}$, so $R(x)$ expresses the relative uncertainty indeed. Also $x = x(u(x) + R(x))$, for $x = xu(x) = x(u(x) + e(u(x)) = x(u(x) + R(x))$.

For precise non-zero numbers the relative uncertainty is equal to 0. In case $x = e(x)$ is a magnitude the formula $R(x) = e(x)x^{-1}$ would amount to $R(x) = e(x)(e(x))^{-1}$, a division by a generalized zero. So it is natural to define the relative uncertainty of a magnitude to be the largest magnitude.

Theorem 4.2. (Distributivity criterion) *Let S be a solid and let $x, y, z \in S$. Then*
 (2) $xy + xz = x(y + z) \Leftrightarrow e(x)(y + z) = e(x)y + e(x)z \vee R(x) \leq R(y) + R(z)$.

By the criterion above, distributivity holds for x if it is true for its magnitude; the only case where it is maybe not true is where y and z are almost opposite, i.e. $y + z$ is much smaller than both y and z . This is easily seen in the most extreme

case. If x is not precise and y and z are precise non-zero with $y = -z$, we have $e(x)(y+z) = 0$ and $e(x)y + e(x)z = e(x)y - e(x)y = e(x)y \neq 0$. If y and z are not precise, distributivity may hold, provided that the relative uncertainty of x is less than or equal to the maximum of the relative uncertainties of y and z , one might say "sharpness cuts"; note that we have already proved in Proposition 3.7 that distributivity always holds if x is precise and in Corollary 2.29 that $e(x)(e(y) + e(z)) = e(x)e(y) + e(x)e(z)$. So the criterion formalizes what was said above: distributivity does not hold in the joint presence of two circumstances: the factor x should be more imprecise than the terms y and z , and these terms should be almost opposite.

The distributivity axiom implies that subdistributivity takes the form of an inequality. This is in line with subdistributivity for interval calculus holding with inclusion.

Theorem 4.3. (Subdistributivity) *Let S be a solid and let $x, y, z \in S$. Then $x(y+z) \leq xy + xz$.*

Proof. Using Proposition 3.4.8 one has

$$x(y+z) \leq x(y+z) + e(x)y + e(x)z = xy + xz.$$

□

We already saw that for distributivity not to hold the two elements y and z of the second member should be in some sense opposite. This can never be if y or z are of the same sign. We show that distributivity effectively holds in this case.

Corollary 4.4. *Let S be a solid and let $x, y, z \in S$. If y and z are of the same sign, then $x(y+z) = xy + xz$.*

Proof. By Theorem 4.2 we only need to show that $e(x)(y+z) = e(x)y + e(x)z$. Suppose firstly that y and z are both positive. By Theorem 4.3 one has $e(x)(y+z) \leq e(x)y + e(x)z$. To show that also $e(x)y + e(x)z \leq e(x)(y+z)$, we assume without loss of generality that $y \leq z$. Then by Axiom 18

$$e(x)y + e(x)z = e(x)z \leq e(x)(z + e(y)) \leq e(x)(y+z).$$

Hence $e(x)(y+z) = e(x)y + e(x)z$.

If y and z are both negative, then $-y$ and $-z$ are positive, hence

$$x(y+z) = -x(-y-z) = -x(-y) - x(-z) = xy + xz.$$

□

Corollary 4.5. *Let S be a solid and let $x, y \in S$. Then $x(y+y) = xy + xy$.*

To prove Theorem 4.2 we distinguish three cases: (i) x is not zeroless, (ii) y or z is not zeroless, (iii) x, y, z are zeroless. In the first case we may assume that y and z are zeroless hence $R(y) + R(z) < R(x)$. Then the criterion states that $e(x)(y+z) = e(x)y + e(x)z$ is equivalent to itself. In Proposition 4.6 we give a more operational version of this criterion. In the second case one always has $R(x) \leq R(y) + R(z)$. Then we must show that $x(z + e(y)) = xz + xe(y)$ always holds; notice that if this particular form of distributivity holds, automatically $R(x) \leq R(e(y)) + R(z)$, because $R(e(y)) = M$. We do so in Proposition 4.7 below. In the third case both conditions of the criterion may happen. Its proof is rather

involved and needs some preliminary results where we take profit of the fact that all elements are zeroless.

Proposition 4.6. *Let S be a solid, $x \in S$ and $y, z \in S^*$. Suppose that $x = e(x)$. Then $e(x)(y+z) = e(x)y + e(x)z$ if and only if $e(x)(y+z) = e(x)y$ or $e(x)(y+z) = e(x)z$.*

Proof. The direct implication follows directly from Axiom 19 and Proposition 2.2.3. For the inverse implication, assume that $e(x)(y+z) = e(x)y$ or $e(x)(y+z) = e(x)z$. Now $e(x)(y+z) \neq e(x)y + e(x)z$ is self-contradictory, for then $e(x)y = e(x)z$ by Lemma 4.9, hence in both cases $e(x)(y+z) = e(x)y + e(x)z$. Hence $e(x)(y+z) = e(x)y + e(x)z$ indeed. \square

Proposition 4.7. *Let S be a solid, $x, y, z \in S$. Then $x(z + e(y)) = xz + xe(y)$.*

Proof. One has $e(x(z + e(y))) = e(x)(z + e(y)) + x(e(z) + e(y))$, by Axiom 20. Then $e(x)z + e(x)e(y) \leq e(x)z + e(x)e(y) + xe(z) + xe(y) = e(x(z + e(y)))$, by Proposition 4.6 and Proposition 2.28. Hence $e(x(z + e(y))) = e(x(z + e(y))) + e(x)z + e(x)e(y)$, by Proposition 2.6. Then $x(z + e(y)) = x(z + e(y)) + e(x)z + e(x)e(y) = xz + xe(y)$, by Axiom 22. \square

We now turn to the third case which is in some sense the generic case. The proof of the necessity part needs some calculatory properties of multiples of magnitudes. We will see that in the presence of the conditions of the criterion for distributivity the correction term $e(x)y + e(x)z$ must be smaller than the magnitude of $x(y+z)$. Combining this fact with subdistributivity will yield distributivity.

Next lemma is a form of cross-multiplication. It is stated as an implication. The converse is much more involved and will be given in Lemma 4.16 below.

Lemma 4.8. *Let S be a solid and let $x, y \in S^*$. If $R(x) \leq R(y)$ then $e(x)y \leq e(y)x$.*

Proof. Assume that $R(x) \leq R(y)$. Then $e(x)x^{-1} = e(u(x)) \leq e(u(y)) = e(y)y^{-1}$. By Proposition 2.17

$$e(x)u(x)y = e(x)x^{-1}xy \leq e(y)y^{-1}xy = e(y)u(y)x.$$

Hence $e(x)y \leq e(y)x$ by Theorem 2.16.1. \square

Lemma 4.9 expresses the fact that when distributivity does not hold for the magnitude of x then y and z must be roughly of the same order of magnitude, i.e. $e(x)y = e(x)z$.

Lemma 4.9. *Let S be a solid and let $x, y, z \in S$. If $e(x)(y+z) \neq e(x)y + e(x)z$, then $e(x)y = e(x)z$.*

Proof. Assume $e(x)(y+z) \neq e(x)y + e(x)z$. Then $e(x)(y+z) < e(x)y + e(x)z$ by Theorem 4.3. We may assume, without loss of generality that $|y| \leq |z|$. Then by Axiom 18

$$(3) \quad e(x)y = e(x)|y| \leq e(x)|z| = e(x)z.$$

As a consequence $e(x)y + e(x)z = e(x)z$ by Proposition 2.6, and we conclude that

$$(4) \quad e(x)(y+z) < e(x)z.$$

In order to show that $e(x)z \leq e(x)y$, from now on we assume that z is positive; the case that z is negative is analogous. We prove that y and z are of opposite

sign and "not too different", in a sense we prove that $y < -z/2$, note that $|y| \leq |z|$ implies the lower bound $-z + e(y) \leq y + e(z)$. The inequality $y + y < -z$ will be obtained by successive approximations. We prove firstly that $y < e(y)$. Suppose towards a contradiction that $e(y) \leq y$. Then by compatibility with addition and Axiom 18 one has $e(x)z \leq e(x)(z + e(y)) \leq e(x)(z + y)$, in contradiction with (4). Hence $y < e(y)$ and $|y| = -y$.

Secondly we show that $e(z) < -y$. Suppose towards a contradiction that $-y \leq e(z)$. Then

$$-y + e(z) \leq e(z) + e(z) = e(z).$$

On the other hand, because $y < e(y)$

$$e(z) \leq e(y) + e(z) \leq -y + e(z).$$

Hence $-y + e(z) = e(z)$. But then

$$\begin{aligned} e(x)(y + z) &= e(x)(-y - z) = e(x)(-y + e(z) - z) \\ &= e(x)(e(z) - z) = e(x)(-z) = e(x)z, \end{aligned}$$

in contradiction with (4). Hence $e(z) < -y$.

We show now that $y + y < -z$. Suppose towards a contradiction that $-z \leq y + y$. Using Axiom 18, Theorem 4.3 and (4) one has

$$\begin{aligned} e(x)z &= e(x)(z + z - z) \leq e(x)(z + z + y + y) \\ &\leq e(x)(y + z) + e(x)(y + z) = e(x)(y + z) < e(x)z, \end{aligned}$$

which is a contradiction. Hence $y + y < -z$, and also $z + e(y) \leq -y - y + e(z)$.

To finish the proof, using the facts that $y + y < -z$ and $e(z) < -y$, Axiom 18 and Theorem 4.3 we see that

$$\begin{aligned} e(x)z &\leq e(x)(z + e(y)) \leq e(x)(-y - y + e(z)) \\ &\leq e(x)(-y - y - y) = e(x)(y + y + y) \\ &\leq e(x)y + e(x)y + e(x)y = e(x)y. \end{aligned}$$

Hence

$$(5) \quad e(x)z \leq e(x)y.$$

Combining (3) and (5) one derives that $e(x)z = e(x)y$. □

Proof of the necessity of the condition for distributivity of Theorem 4.2. We need only to consider the zeroless case. Assume firstly that $e(x)(y + z) = e(x)y + e(x)z$. One has

$$\begin{aligned} xy + xz &= x(y + z) + e(x)y + e(x)z \\ &= x(y + z) + e(x(y + z)) + e(x)(y + z) \\ &= x(y + z) + xe(y + z) + e(x)(y + z) + e(x)(y + z) \\ &= x(y + z) + xe(y + z) + e(x)(y + z) \\ &= x(y + z) + e(x(y + z)) = x(y + z). \end{aligned}$$

Secondly, assume that $R(x) \leq R(y) + R(z)$, where we may suppose that $e(x)(y + z) < e(x)y + e(x)z$. Then $e(x)y = e(x)z$ by Lemma 4.9 and $e(x)y \leq e(y)x$ by Lemma

4.8. Hence, using Proposition 2.28 and Axiom 20

$$\begin{aligned}
 x(y+z) &= x(y+z) + e(x(y+z)) \\
 &= x(y+z) + e(x)(y+z) + xe(y) + xe(z) \\
 &= x(y+z) + e(x)(y+z) + xe(y) + xe(z) + e(x)y \\
 &= x(y+z) + xe(y+z) + e(x)y \\
 &= x(y+z) + e(x)y + e(x)z \\
 &= xy + xz.
 \end{aligned}$$

□

We now turn to the sufficiency of this condition for distributivity of Theorem 4.2. For the zeroless case, the converse of Lemma 4.8 is needed. The proof of this converse uses two important properties of the unity element. First, one has $e(x)u(y) = e(x)$ for arbitrary x, y , generalizing Theorem 2.16.1, and second the decomposition $u(x) = 1 + e(u(x))$.

Remark 4.10. *By Axiom 28 there is a precise element b such that $u(x) - 1 = b + e(u(x) - 1) = b + e(u(x))$. So one can write $u(x) = 1 + b + e(u(x))$ and we must show that b can be chosen equal to 0.*

We prove first some preliminary results. If an element x is zeroless, the absolute value of the precise part of its decomposition must be larger than its magnitude. Also, dividing the magnitude of x by x itself is the same as dividing the magnitude by any of its precise parts.

Proposition 4.11. *Let S be a solid. Let $x = a + e(x) \in S^*$ be such that $e(a) = 0$. Then*

- (1) $e(x) < |a|$.
- (2) $e(x)a^{-1} = e(x)x^{-1} = e(u(x))$.

Proof. 1. Assume firstly that $e(x) < x$. Suppose that $a \leq e(x)$. Then by compatibility with addition $x = a + e(x) \leq e(x) + e(x) = e(x)$, which is a contradiction. Then $0 < a$ by Proposition 2.8.2. Hence $e(x) < a = |a|$.

Assume secondly that $x < e(x)$. Then $a + e(x) < e(x)$, hence $e(x) < -a$ by compatibility with addition. By transitivity $0 < -a$, so $-a = |a|$. Hence $e(x) < |a|$.

2. By Part 1 one has $0 \leq e(x) < |a|$. We start by proving that $e(x)a^{-1} \leq e(u(x))$. Assume firstly $0 < a$. Now $a \leq a + e(x) = x$. Hence $1 = aa^{-1} \leq xa^{-1}$. Then by Axiom 18 and Theorem 2.16.1

$$(6) \quad e(u(x)) = e(x)x^{-1} = e(x)x^{-1} \cdot 1 \leq e(x)u(x)a^{-1} = e(x)a^{-1}.$$

Secondly, assume $a < 0$. Then $0 < -a$ and $0 < (-a)^{-1}$ by Proposition 2.24. Moreover, $-a \leq -a + e(x) = -x$ by Proposition 3.4.8. Hence $1 = (-a)(-a)^{-1} \leq (-x)(-a)^{-1} = xa^{-1}$. Then we derive (6) as above.

We prove now that $e(x)a^{-1} \leq e(u(x))$. Using Theorem 2.16.1 and Proposition 3.9 one derives

$$\begin{aligned}
 e(x)a^{-1} &= e(x)u(x)a^{-1} = e(x)x^{-1}(a + e(x))a^{-1} \\
 &= e(x)x^{-1}(1 + e(x)a^{-1}).
 \end{aligned}$$

Then by Part 1

$$e(x) a^{-1} \leq e(x) x^{-1} (1 + aa^{-1}) = e(x) x^{-1} (1 + 1).$$

Hence by Theorem 4.3

$$(7) \quad e(x) a^{-1} \leq e(x) x^{-1} + x^{-1} e(x) = x^{-1} e(x) = e(u(x)).$$

The result follows from (6) and (7). \square

Using the previous proposition one shows that the element b of Remark 4.10 has to be "small" in the sense that the absolute value of b is less than or equal to the magnitude of the unity of x .

Lemma 4.12. *Let S be a solid. Let $x = a + e(x) \in S^*$. Suppose $u(x) = 1 + b + e(u(x))$. Then $|b| \leq e(u(x))$.*

Proof. Because $R(e(x)) = M = R(e(u(x)))$ and $R(a) = 0$, one derives

$$\begin{aligned} x &= xu(x) = (a + e(x))(1 + b + e(u(x))) \\ &= (a + e(x))(1 + b) + (a + e(x))e(u(x)) \\ &= a(1 + b) + e(x)(1 + b) + xe(u(x)). \end{aligned}$$

Then, because a is precise,

$$a + e(x) = a + ab + e(x)(1 + b) + e(x).$$

Hence

$$e(x) = ab + e(x)(1 + b) + e(x).$$

Now Proposition 2.6 implies that $e(x)(1 + b) \leq e(x)$, for

$$\begin{aligned} e(x) &= e(e(x)) = e(ab) + e(e(x)(1 + b)) + e(e(x)) \\ &= e(x) + e(x)(1 + b). \end{aligned}$$

Hence $e(x) = ab + e(x)$. Then $|ab| \leq e(x)$. Hence $|b| \leq e(x) a^{-1} = e(u(x))$ by Proposition 4.11.2. \square

We now show that a unity can be written as the sum of the precise unity 1 and the imprecision of the unity.

Theorem 4.13. (Expansion of unity) *Let S be a solid and let $x \in S^*$. Then $u(x) = 1 + e(u(x))$.*

Proof. By Lemma 4.12 one may suppose $u(x) = 1 + b + e(u(x))$ with $|b| \leq e(u(x))$. This means that $b + e(u(x)) = e(u(x))$. Hence $u(x) = 1 + e(u(x))$. \square

Corollary 4.14. *Let S be a solid and let $x \in S^*$. Then $e(u(x)) < 1$.*

Ordered fields satisfy the property that $0 < 1$ and $1 \cdot 0 = 0$. We generalized these properties to $e(u(x)) < u(x)$ and $e(x)u(x) = e(x)$. Within solids stronger properties are valid. Indeed, if x and y are arbitrary elements of a solid, then $e(u(x)) < 1$ and $e(x)u(y) = e(x)$.

Theorem 4.15. *Let S be a solid. Let $x \in S$ and $y \in S^*$. Then $e(x)u(y) = e(x)$.*

Proof. By Theorem 4.13

$$e(x)u(y) = e(x)(1 + e(u(y))).$$

Then by Corollary 2.31

$$e(x)u(y) = e(x)1 + e(x)e(u(y)) = e(x) + e(x)e(u(y)).$$

By Corollary 4.14 and Axiom 18 one has $e(x)e(u(y)) \leq e(x)$. Hence $e(x)u(y) = e(x)$ by Proposition 2.6. \square

We are now able to derive the converse to Lemma 4.8.

Lemma 4.16. *Let S be a solid and let $x, y \in S^*$. If $e(x)y \leq e(y)x$ then $R(x) \leq R(y)$.*

Proof. Suppose that $e(x)y \leq e(y)x$. Without restriction of generality we may assume that x and y are positive. Then $e(x)u(y)x^{-1} \leq e(y)y^{-1}u(x)$ by Axiom 17. By Theorem 4.15 one has $e(x)x^{-1} \leq e(y)y^{-1}$. Hence $R(x) \leq R(y)$. \square

Proof of the sufficiency of the condition for distributivity of Theorem 4.2. We only need to consider the zeroless case. Assume that $xy + xz = x(y + z)$. With this equality, applying cancellation to Axiom 22 gives

$$e(x(y + z)) = e(x(y + z)) + e(x)y + e(x)z.$$

Then

$$(8) \quad e(x)y + e(x)z \leq e(x(y + z)) = e(x)(y + z) + xe(y) + xe(z).$$

We consider three cases: (i) $e(x)y + e(x)z \leq e(x)(y + z)$, and if (i) does not hold, (ii) $e(x)y + e(x)z \leq xe(y)$ and (iii) $e(x)y + e(x)z \leq xe(z)$.

(i) One has $e(x)y + e(x)z = e(x)(y + z)$ by (8) and Theorem 4.3.

(ii) Lemma 4.9 implies that $e(x)y = e(x)z$. Then $e(x)y \leq xe(y)$. Hence $R(x) \leq R(y) \leq R(y) + R(z)$ by Lemma 4.16.

(iii) This case is analogous to case (ii). \square

This completes the proof of Theorem 4.2.

4.2. Equivalent form of the distributivity axiom. We show that the distributivity condition of Axiom 22 and formula (2) of Theorem 4.2 are equivalent. In order to do so we need a property of subdistributivity as well as some special cases of the distributive law. All these properties are supposed to be shown without using Axiom 22, but in the presence of (2).

Proposition 4.17. *Let $x, y, z \in S$. Suppose (2) holds. Then $e(x)(y + z) \leq e(x)y + e(x)z$.*

The proposition is equivalent to Axiom 22 in case $x = e(x)$, for then it takes the form

$$e(x)y + e(x)z = e(x)(y + z) + e(x)y + e(x)z.$$

We now present the special cases of the distributive law needed to prove the proposition.

Lemma 4.18. *Let $x, y, z \in S$. If x is precise, then $x(y + z) = xy + xz$.*

Proof. By Proposition 3.5 one has $e(x)(y+z) = 0(y+z) = 0$ and $e(x)y + e(x)z = 0 + 0 = 0$. Hence $e(x)(y+z) = e(x)y + e(x)z$ and one concludes that $x(y+z) = xy + xz$ by (2). \square

Lemma 4.19. *Let $x \in S$. Let $p \in S$ be precise. Then $e(x)(p+p) = e(x)p$.*

Proof. By Axiom 18 one has $e(x) \leq e(x)(1+1)$. Suppose that $e(x) < e(x)(1+1)$. By Axiom 29 there exists a precise element q such that $e(x) < q < e(x)(1+1)$. Then $e(x) < q/(1+1)$, otherwise by Lemma 4.18 one would have $q = q/(1+1) + q/(1+1) \leq e(x) + e(x) = e(x)$. Hence $e(x)(1+1) \leq q$, a contradiction. Then $e(x)(1+1) \leq e(x)$ and one concludes that $e(x)(1+1) = e(x)$. Then $e(x)(p+p) = e(x)p$ by Lemma 4.18. \square

Lemma 4.20. *Let $x, y \in S$. Let $p \in S$ be precise. Then $x(p + e(y)) = xp + xe(y)$.*

Proof. Immediately from (2), for $R(x) \leq R(p) + R(e(y)) = M$. \square

Proof of Proposition 4.17. Suppose without loss of generality that $|y| \leq |z|$. If y and z have opposite signs then

$$e(x)(y+z) \leq e(x)z \leq e(x)y + e(x)z.$$

If y and z have the same sign we may assume that they are both positive by Proposition 2.13. Let $z = p + e(z)$. Then by Lemma 4.20 and Lemma 4.19

$$\begin{aligned} e(x)(y+z) &\leq e(x)(z+z) \\ &= e(x)((p+p) + e(z)) \\ &= e(x)(p+p) + e(x)e(z) \\ &= e(x)p + e(x)e(z) = e(x)z \leq e(x)y + e(x)z. \end{aligned}$$

\square

Theorem 4.21. *Let $x, y, z \in S$. Then (2) holds if and only if $xy + xz = x(y+z) + e(x)y + e(x)z$.*

Proof. By Theorem 4.2 we only need to prove the necessary part. Suppose (2) holds. By Axiom 28 there is a precise number a such that $x = a + e(x)$. By Lemma 4.20

$$x(y+z) + e(x)y + e(x)z = a(y+z) + e(x)(y+z) + e(x)y + e(x)z.$$

By Proposition 4.17 and Lemma 4.18

$$\begin{aligned} x(y+z) + e(x)y + e(x)z &= a(y+z) + e(x)y + e(x)z \\ &= ay + az + e(x)y + e(x)z. \end{aligned}$$

Then by Lemma 4.20

$$x(y+z) + e(x)y + e(x)z = (a + e(x))y + (a + e(x))z = xy + xz.$$

\square

APPENDIX: LIST OF AXIOMS

(1) **Axioms for addition****Axiom 1.** $\forall x \forall y \forall z (x + (y + z) = (x + y) + z).$ **Axiom 2.** $\forall x \forall y (x + y = y + x).$ **Axiom 3.** $\forall x \exists e (x + e = x \wedge \forall f (x + f = x \rightarrow e + f = e)).$ **Axiom 4.** $\forall x \exists s (x + s = e(x) \wedge e(s) = e(x)).$ **Axiom 5.** $\forall x \forall y (e(x + y) = e(x) \vee e(x + y) = e(y)).$ (2) **Axioms for multiplication****Axiom 6.** $\forall x \forall y \forall z (x(yz) = (xy)z).$ **Axiom 7.** $\forall x \forall y (xy = yx).$ **Axiom 8.** $\forall x \neq e(x) \exists u (xu = x \wedge \forall v (xv = x \rightarrow uv = u)).$ **Axiom 9.** $\forall x \neq e(x) \exists d (xd = u(x) \wedge u(d) = u(x)).$ **Axiom 10.** $\forall x \neq e(x) \forall y \neq e(y) (u(xy) = u(x) \vee u(xy) = u(y)).$ (3) **Order axioms****Axiom 11.** $\forall x (x \leq x).$ **Axiom 12.** $\forall x \forall y (x \leq y \wedge y \leq x \rightarrow x = y).$ **Axiom 13.** $\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z).$ **Axiom 14.** $\forall x \forall y (x \leq y \vee y \leq x).$ **Axiom 15.** $\forall x \forall y \forall z (x \leq y \rightarrow x + z \leq y + z).$ **Axiom 16.** $\forall x \forall y (y + e(x) = e(x) \rightarrow (y \leq e(x) \wedge -y \leq e(x))).$ **Axiom 17.** $\forall x \forall y \forall z ((e(x) < x \wedge y \leq z) \rightarrow xy \leq xz).$ **Axiom 18.** $\forall x \forall y \forall z ((e(y) \leq y \leq z) \rightarrow e(x)y \leq e(x)z).$ (4) **Axioms relating addition and multiplication****Axiom 19.** $\forall x \forall y \exists z (e(x)y = e(z)).$ **Axiom 20.** $\forall x \forall y (e(xy) = e(x)y + e(y)x).$ **Axiom 21.** $\forall x \neq e(x) (e(u(x)) = e(x)/x).$ **Axiom 22.** $\forall x \forall y \forall z (xy + xz = x(y + z) + e(x)y + e(x)z).$ **Axiom 23.** $\forall x \forall y (-(xy) = (-x)y).$ (5) **Axioms of existence****Axiom 24.** $\exists m \forall x (m + x = x).$ **Axiom 25.** $\exists u \forall x (ux = x).$ **Axiom 26.** $\exists M \forall x (e(x) + M = M).$ **Axiom 27.** $\exists x (e(x) \neq 0 \wedge e(x) \neq M).$ **Axiom 28.** $\forall x \exists a (x = a + e(x) \wedge e(a) = 0).$ **Axiom 29.** $\forall x \forall y (x = e(x) \wedge y = e(y) \wedge x < y \rightarrow \exists z (z \neq e(z) \wedge x < z < y)).$

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